

GENERALIZED STRETCHED IDEALS AND SALLY'S CONJECTURE

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ABSTRACT. We introduce the concept of j -stretched ideals in a Noetherian local ring. This notion generalizes to arbitrary ideals the classical notion of stretched \mathfrak{m} -primary ideals of Sally and Rossi-Valla, as well as the concept of ideals of minimal and almost minimal j -multiplicity introduced by Polini-Xie. One of our main theorems states that, for a j -stretched ideal, the associated graded ring is Cohen-Macaulay if and only if two classical invariants of the ideal, the reduction number and the index of nilpotency, are equal. Our second main theorem, presenting numerical conditions which ensure the almost Cohen-Macaulayness of the associated graded ring of a j -stretched ideal, provides a generalized version of Sally's conjecture. This work, which also holds for modules, unifies the approaches of Rossi-Valla and Polini-Xie and generalizes simultaneously results on the Cohen-Macaulayness or almost Cohen-Macaulayness of the associated graded module by several authors, including Sally, Rossi-Valla, Wang, Elias, Corso-Polini-Vaz Pinto, Huckaba, Marley and Polini-Xie.

1. INTRODUCTION

Given a Noetherian local ring (R, \mathfrak{m}) and an ideal I of R , it is well-known that the associated graded ring $\mathrm{gr}_I(R) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ encodes algebraic and geometric properties of I . Indeed, $\mathrm{Proj}(\mathrm{gr}_I(R))$ is the exceptional fiber of the blow-up of $\mathrm{Spec}(R)$ along the subvariety $V(I)$. Strong efforts have been given in the last thirty years to detect conditions on R and I which guarantee that $\mathrm{gr}_I(R)$ has sufficiently high depth (more precisely, $\mathrm{gr}_I(R)$ being Cohen-Macaulay or almost Cohen-Macaulay), due to the reason that high depth of the associated graded ring forces the vanishing of its cohomology groups and thereby allows one to compute, or bound, relevant numerical invariants such as the Castelnuovo-Mumford regularity or the number and degrees of the defining equations of the blow-up (see for instance, [15] and [14]).

The classical method, originated from the pioneering work of Sally, studies the interplay between the Hilbert coefficients of an \mathfrak{m} -primary ideal and the depth of the associated graded ring. The idea is that extremal values of the Hilbert coefficients yield high depth of the associated graded ring and, conversely, good depth properties encode all the information about the Hilbert function.

In 1967, Abhyankar proved that the multiplicity of a d -dimensional Cohen-Macaulay local ring (R, \mathfrak{m}) can be written as $e_0(\mathfrak{m}) = \mu(\mathfrak{m}) - d + K$ for some integer $K \geq 1$, where $\mu(\mathfrak{m})$ is the embedding dimension of R [2]. Since then, rings for which $e_0(\mathfrak{m}) = \mu(\mathfrak{m}) - d + 1$ (respectively, $e_0(\mathfrak{m}) = \mu(\mathfrak{m}) - d + 2$) have been called *rings of minimal multiplicity* (respectively, *rings of almost minimal multiplicity*). These notions were extended by Sally to stretched Cohen-Macaulay local rings by requiring an Artinian reduction R/J , where J is a minimal reduction of \mathfrak{m} , to be

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stretched, i.e., the ideal $(\mathfrak{m}/J)^2$ is a principal ideal (see [26] and [23]). Sally studied the Cohen-Macaulay and almost Cohen-Macaulay property of the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ for those classes of rings. She proved that $\mathrm{gr}_{\mathfrak{m}}(R)$ is always Cohen-Macaulay if R has minimal multiplicity [25]. Unfortunately, for arbitrary Cohen-Macaulay local rings of almost minimal multiplicity (as well as stretched Cohen-Macaulay local rings), the Cohen-Macaulay property of $\mathrm{gr}_{\mathfrak{m}}(R)$ fails to hold [27]. However, Sally conjectured that if R has almost minimal multiplicity then $\mathrm{gr}_{\mathfrak{m}}(R)$ is almost Cohen-Macaulay. This conjecture was proved thirteen years later by Rossi and Valla [21], and, independently, by Wang [32]. Later, in 2001, Rossi and Valla extended the notion of stretched Cohen-Macaulay local rings of Sally to stretched \mathfrak{m} -primary ideals, and proved an extended version of Sally's conjecture by giving conditions for the associated graded rings of stretched \mathfrak{m} -primary ideals to be almost Cohen-Macaulay [23].

During the last twenty years, another method has also been developed to study the depth of the associated graded rings of general ideals (see [11], [28], [7], [8], [9], [14], [6], [1], and related papers). Essentially, this method requires the ideal I to have certain residual intersection properties (automatically satisfied if I is \mathfrak{m} -primary) and sufficiently many powers of I to have high depth, where the number of powers of I required to have high depth depends on the reduction number of I . Since the depth drops dramatically for higher powers of I , this method works well if I has “relatively small” reduction number.

Recently, Polini and Xie [18] proved Sally's conjecture for a class of ideals that are not necessarily \mathfrak{m} -primary by combining the techniques of \mathfrak{m} -primary ideals with tools from residual intersections. They extended the notions of minimal and almost minimal multiplicity to arbitrary ideals by defining the concepts of minimal and almost minimal j -multiplicity, and proved that, under certain residual assumptions, the associated graded ring is Cohen-Macaulay (respectively, almost Cohen-Macaulay) for ideals having minimal j -multiplicity (respectively, almost minimal j -multiplicity).

In the present paper, we propose a more general numerical condition on I that extends the classical stretched \mathfrak{m} -primary ideals defined by Sally, Rossi and Valla, as well as the minimal and almost minimal j -multiplicity introduced by Polini and Xie. Let (R, \mathfrak{m}) be a Noetherian local ring of dimension d with infinite residue field (we can enlarge the residue field to be infinite by replacing R by $R(z) = R[z]_{\mathfrak{m}R[z]}$, where z is a variable over R). Let I be an R -ideal of maximal analytic spread. Recall that the quotient ring of R modulo a general $d - 1$ -geometric residual intersection of I is a 1-dimensional Noetherian local ring and the ideal generated by the image of I in this quotient ring is primary to its maximal ideal (thus it allows us to reduce to the setting of the classical \mathfrak{m} -primary case). Roughly speaking, the ideal I is j -stretched if it generates a stretched \mathfrak{m} -primary ideal (in the sense of Rossi and Valla) after reducing to this 1-dimensional Noetherian local ring. Since j -stretched ideals are not necessarily \mathfrak{m} -primary, to study them, we adopt the tools of general elements, residual intersection theory (a generalization of linkage), and the notion of j -multiplicity

(introduced by Archilles and Manaresi as a higher dimensional version of the Hilbert multiplicity [3]). We refer to Section 2 in the following for a more detailed elaboration of j -stretched ideals.

One of the most important features of \mathfrak{m} -primary ideals I comes from the fact that they have finite colength $\lambda(R/I)$, which makes many tools and computations applicable. When I is arbitrary, one would like to reduce to the case of finite colength by factoring out a sequence of elements. But the problem is that the colength depends on the choice of a sequence of elements. To overcome this difficult, we develop a “Specialization Lemma” (see Lemma 3.1 in Section 3) stating that, if we choose a sequence of general elements, we will have a fixed colength. Moreover, if R is equicharacteristic, general specializations yield the smallest colength. We apply this lemma to study the index of nilpotency and the stretchedness property. For instance, we generalize to non \mathfrak{m} -primary ideals I a proposition proved by Fouli [5, Proposition 5.3.3], stating that, over an equicharacteristic Cohen-Macaulay local ring, the index of nilpotency of I does not depend on the general minimal reduction, and general minimal reductions always achieve the largest possible index of nilpotency. We also answer a question of Sally (see [26]) asking: to what extent does the classical notion of stretchedness depend on minimal reductions? As a consequence of Lemma 3.1, one obtains the answer that the stretchedness property does not depend on the choice of a general minimal reductions. We remark here that Lemma 3.1 may be of independent interest to the reader, as it can also be interpreted as an upper-semicontinuity result of lengths.

We now state our main theorems. For any j -stretched ideal I with certain residual intersection properties (automatically satisfied if I is \mathfrak{m} -primary), we prove in Theorem 4.1 that the associated graded ring $\text{gr}_I(R)$ is Cohen-Macaulay if and only if the reduction number of I and its index of nilpotency coincide. The second main result, Theorem 4.6, provides a sufficient condition for the associated graded rings of j -stretched ideals to be almost Cohen-Macaulay and is a generalized version of Sally's conjecture. Our criteria are purely numerical and could be applied to ideals with arbitrarily large reduction numbers. Indeed, we provide a class of j -stretched ideals having arbitrarily large reduction number such that the Cohen-Macaulay property of the associated graded ring follows from our main theorem, but from no previous result in the literature (see Example 4.3 in Section 4).

The structure of the paper is the following: In Section 2, we define the concept of j -stretched ideals and recall definitions of residual intersections. Section 3 is rather technical and includes the Specialization Lemma (Lemma 3.1) as well as several results on the structure of j -stretched ideals. Section 4 contains our two main theorems, giving numerical characterizations of the Cohen-Macaulayness and almost Cohen-Macaulayness of the associated graded rings of j -stretched ideals (Theorem 4.1 and Theorem 4.6). Among the applications of these theorems, we recover the main results of [18] and [23], and prove, under additional assumptions, that the associated graded rings of ideals having almost-almost minimal j -multiplicity are almost Cohen-Macaulay (Corollary 4.11).

Finally, in Section 5, we prove the non-trivial fact that j -stretched ideals do generalize stretched \mathfrak{m} -primary ideals (Theorem 5.3 and Corollary 5.4). Although in general these two notions are different, we provide a sufficient condition for them to coincide (Proposition 5.5). As an application, we answer a question raised by Sally (Corollary 5.6).

For the sake of clarity, we will only focus on the case of associated graded rings $\text{gr}_I(R)$, although all the definitions and results can be extended and proved for associated graded modules $\text{gr}_I(M)$, where M is a finite module over R .

2. THE MAIN DEFINITIONS

In this section we fix the notation, introduce j -stretched ideals and recall some definitions and facts from residual intersection theory.

Throughout this paper, we always assume that (R, \mathfrak{m}, k) is a Noetherian local ring of dimension d with maximal ideal \mathfrak{m} and infinite residue field $k = R/\mathfrak{m}$ (possibly, after enlarging the residue field k).

- The *associated graded ring* of an R -ideal I is defined as $G = \text{gr}_I(R) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$.
- An ideal $J \subseteq I$ is called a *reduction* of I if there exists a non-negative integer r such that $I^{r+1} = JI^r$. The least r such that $I^{r+1} = JI^r$ is denoted by $r_J(I)$, and called the *reduction number of I with respect to J* .
- A reduction is called *minimal* if it is minimal with respect to inclusion.
- The *reduction number* $r(I)$ of I is defined as $\min\{r_J(I) \mid J \text{ a minimal reduction of } I\}$.
- Finally, since $|k| = \infty$, minimal reductions of I always exist, and every minimal reduction of I can be minimally generated by the same number of generators, $\ell(I)$, dubbed the *analytic spread* of I . Since the inequality $\ell(I) \leq d = \dim R$ always holds, one says that I has *maximal analytic spread* if $\ell(I) = d$.

Write $I = (a_1, \dots, a_s)$ and $x_i = \sum_{j=1}^s \lambda_{ij} a_j$ for $i = 1, \dots, t$ and $(\lambda_{ij}) \in R^{ts}$. The elements x_1, \dots, x_t are *general* in I if there exists a Zarisky dense open subset U of k^{ts} such that $(\overline{\lambda_{ij}}) \in U$, where $\overline{}$ denotes images in the residue field k . The relevance of this notion in our analysis comes from the following facts:

- (a) General elements in I always form a superficial sequence for I ([33, Corollary 2.5]);
- (b) If $t = \ell(I)$ then a sequence x_1, \dots, x_t of general elements in I forms a minimal reduction of I with reduction number $r(I)$ (see for instance [29, Corollary 2.2]);
- (c) One can use general elements to compute the j -multiplicity of the ideal I ([18, Proposition 2.1]).

Notation. From now on, we assume I has maximal analytic spread $\ell(I) = d$, and J is a *general* minimal reduction of I , i.e., $J = (x_1, \dots, x_d)$, where x_1, \dots, x_d are d general elements in I .

We write $\overline{R} = R/J_{d-1} : I^\infty$, where $J_{d-1} : I^\infty = \{b \in R \mid \exists \delta > 0 \text{ such that } b \cdot I^\delta \subseteq J_{d-1}\}$ and $J_{d-1} = (x_1, \dots, x_{d-1})$. We use $\overline{}$ to denote images in the quotient ring \overline{R} .

Note that $\overline{R} \neq 0$ if and only if $\ell(I) = d$ [17]. Indeed in this case \overline{R} is an 1-dimensional Cohen-Macaulay local ring and \overline{I} is primary to the maximal ideal $\overline{\mathfrak{m}}$.

Therefore, one can define the Hilbert function of I on \overline{R} :

$$HF_{I, \overline{R}}(n) = \lambda(\overline{I}^n / \overline{I}^{n+1}), \text{ for } n \geq 0,$$

which is independent of a choice of the general minimal reduction J (by Lemma 3.1 in Section 3, or see [19]). The j -multiplicity of I is computed as follows (see for instance [18, Proposition 2.1])

$$j(I) = e(I, \overline{R}) = \lambda(\overline{R}/x_d \overline{R}) = \lambda(\overline{I}/x_d \overline{I}) = \lambda(\overline{I}/\overline{I}^2) + \lambda(\overline{I}^2/x_d \overline{I}).$$

We are now ready to give the definition of j -stretched ideals.

Definition 2.1. Let R , I and J be the same as above. We say that I is j -stretched if

$$\lambda(\overline{I}^2/x_d \overline{I} + \overline{I}^3) \leq 1.$$

Observe that if I is a j -stretched ideal then the Artinian reduction $\overline{R}/(\overline{x}_d)$ possesses a stretched Hilbert function with respect to I , i.e.,

$$H_{\overline{I}/(\overline{x}_d)}(2) = \lambda(\overline{I}^2/(\overline{x}_d) \cap \overline{I}^2 + \overline{I}^3) \leq \lambda(\overline{I}^2/x_d \overline{I} + \overline{I}^3) \leq 1.$$

Furthermore, if I has *minimal j -multiplicity* (respectively, *almost minimal j -multiplicity*), i.e., the length $\lambda(\overline{I}^2/x_d \overline{I}) = 0$ (respectively, $\lambda(\overline{I}^2/x_d \overline{I}) \leq 1$) (see [18]), then it is easy to see that I is j -stretched; hence the notion of j -stretched ideals includes ideals having minimal or almost minimal j -multiplicity. In particular, every \mathfrak{m} -primary ideal having minimal or almost minimal multiplicity is j -stretched. We will see in Section 5 that j -stretched ideals also generalize stretched \mathfrak{m} -primary ideals (Corollary 5.4).

The property of j -stretchedness is preserved under faithfully flat ring extensions. Indeed let (S, \mathfrak{n}) be a Noetherian local ring that is flat over R with $\mathfrak{m}S = \mathfrak{n}$. If I is j -stretched then IS is a j -stretched ideal of S . Therefore the property of being j -stretched still holds after passing to the completion of R , or enlarging the residue field.

We now recall some definitions and facts from the theory of residual intersections (see for instance [30], [14] and [18]), which will be used frequently in the rest of the paper.

- An ideal I has the G_t condition if $I_{\mathfrak{p}}$ can be generated by i elements for every $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} = i < t$.
- Let $H_t = (x_1, \dots, x_t)$, where x_1, \dots, x_t are elements in I . Define $H_t : I = \{b \in R \mid b \cdot I \subseteq H_t\}$. One says that $H_t : I$ is a t -residual intersection of I if $I_{\mathfrak{p}} = (x_1, \dots, x_t)_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq t - 1$.
- A t -residual intersection $H_t : I$ is called a *geometric t -residual intersection* of I if, in addition, $I_{\mathfrak{p}} = (x_1, \dots, x_t)_{\mathfrak{p}}$ for every $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq t$.
- It is well-known that, if I satisfies the G_t condition, then for general elements x_1, \dots, x_t in I and each $0 \leq i < t$, the ideal $H_i : I$ is a geometric i -residual intersection of I , and $H_t : I$ is a t -residual intersection of I (see [30] and [18, Lemma 3.1]).

- Finally, let R be Cohen-Macaulay, the ideal I has the *Artin-Nagata property* AN_t^- if, for every $0 \leq i \leq t$ and every geometric i -residual intersection $H_i : I$ of I , one has that $R/H_i : I$ is Cohen-Macaulay [30].

Assume R is Cohen-Macaulay. We now list a few classes of ideals satisfying the above residual properties.

- (★) The properties G_d and AN_{d-2}^- are automatically satisfied by any \mathfrak{m} -primary ideal of R .
- (★) Assume $\dim(R/I) = 1$. Then the property AN_{d-2}^- is trivially satisfied by I . Furthermore, I has the G_d condition if and only if I is generically a complete intersection.
- (★) Recall that I is *strongly Cohen-Macaulay* if all of the Koszul homology modules with respect to a generating set of I are Cohen-Macaulay R/I -modules. The property AN_{d-2}^- is satisfied by any strongly Cohen-Macaulay ideal I which satisfies the G_d condition [13]. Examples of strongly Cohen-Macaulay ideals are complete intersections and, if R is Gorenstein, any *licci* ideal I , meaning that I is in the linkage class of a complete intersection, which generalizes the classes of perfect ideals of grade two and Gorenstein ideals of grade three [12].
- (★) Assume R is Gorenstein. Then by linkage theory, the property AN_{d-2}^- is satisfied by any Cohen-Macaulay ideal I with $\dim(R/I) = 2$, and, more generally, by any licci ideal I (for instance, this follows by the above, the facts that the deformation of a licci ideal is licci, and any licci ideal I has a deformation that has the G_d property, and [30, Lemma 1.13]).

We now provide examples of j -stretched ideals in Noetherian local rings.

Example 2.2. Fix $r \geq 1$. Let $R = \mathbb{C}[[x, y, z]]/(x, y) \cap (x^{r+1}, z) = \mathbb{C}[[x, y, z]]/(x^{r+1}, xz, yz)$ and $I = (x, y)$. Then R is an 1-dimensional Cohen-Macaulay local ring and I is a Cohen-Macaulay prime ideal that has $\ell(I) = 1$, G_1 condition and AN_{d-2}^- (automatically satisfied since $d = 1$). Furthermore, I is j -stretched with reduction number r . If $r > 2$ then I does not have almost minimal j -multiplicity.

Proof. It is easy to see that R is an 1-dimensional Cohen-Macaulay local ring and I is a Cohen-Macaulay prime ideal that has $\ell(I) = 1$, G_1 condition and AN_{d-2}^- . We only need to show that I is j -stretched with reduction number r . First notice that $\overline{R} = R/0 : I^\infty = R/0 : I = R/(x^{r+1}, z) \cong k[[x, y]]/(x^{r+1})$, and use $\overline{}$ to denote images in the quotient ring \overline{R} . Let $f = \alpha x + \beta y$ be a general element in I . Then $J = (f)$ is a minimal reduction of I , hence $\beta \neq 0$ since otherwise $I/(f)$ can not have finite length. Replacing y by f , we may assume that $f = y$ and the length

$$\lambda(\overline{I^2}/f\overline{I} + \overline{I^3}) = \lambda((x, y)^2/(y(x, y) + (x, y)^3 + (x^{r+1}))) = \lambda((x, y)^2/(y^2, xy, x^{r+1})) \leq 1,$$

proving the j -stretchedness of I . Notice that $y(x, y)^{r-1} + (x^{r+1}) \subsetneq (x, y)^r$ and $y(x, y)^r + (x^{r+1}) = (x, y)^{r+1}$, hence $r_{(\overline{y})}(\overline{I}) = r$. Since $[0 : I] \cap I = 0$, the equality $(\overline{y})\overline{I}^r = \overline{I}^{r+1}$ implies that

$$I^{r+1} \subseteq yI^r + [0 : I] \cap I^{r+1} = yI^r,$$

which gives $r_{(y)}(I) \leq r$. Our desired result follows since $r_{(y)}(I) \geq r_{(\overline{y})}(\overline{I}) = r$.

Finally, since $\lambda(\overline{I^2}/f\overline{I}) = \lambda[(x, y)^2/(y(x, y) + (x^{r+1}))] = r - 1$, the ideal I does not have almost minimal j -multiplicity if $r > 2$. \square

Example 2.3. Fix $r \geq 1$. Let $R = k[[x, y, z]]/(x^r - yz, y^r - xz, xyz) \cap (x^{r+1} - y^{r+1}, z)$ and $I = (x, y)$. Then R is an 1-dimensional Noetherian local ring (not Cohen-Macaulay) and I is a Cohen-Macaulay prime ideal that has $\ell(I) = 1$, G_1 condition and AN_{d-2}^- (automatically satisfied since $d = 1$). Furthermore, I is j -stretched with reduction number r . Write $\overline{R} = R/0 : I^\infty = R/(x^{r+1} - y^{r+1}, z) \cong k[[x, y]]/(x^{r+1} - y^{r+1})$. One has $\lambda(\overline{I^t}/J\overline{I^{t-1}} + \overline{I^{t+1}}) = 1$ for all $2 \leq t \leq r$ and a general minimal reduction J of I . This implies that I does not have almost minimal j -multiplicity if $r > 2$.

We now exhibit monomial ideals and ideals of points in \mathbb{P}^N that are j -stretched.

Example 2.4. Assume I is the defining ideal of either (i) a set of $n = 6$ general points in \mathbb{P}^2 , or (ii) a set of $n = 4$ or $n = 5$ general points in \mathbb{P}^3 . Then I is a j -stretched Cohen-Macaulay ideal which is generated in a single degree, has $\ell(I) = 1$, G_1 condition and AN_{-1}^- .

Example 2.5. Let I be either the ideal $(a^2b^2, a^2c^2, abc^2, b^3c)$ or (a^3, a^2b, b^2c, ac^2) in $R = k[a, b, c]$. Then I is a height 2 ideal that is not unmixed (indeed, the maximal ideal is an associated prime ideal of I). Computations show that $\lambda(\overline{I^t}/x_3\overline{I^{t-1}} + \overline{I^{t+1}}) = 1$ for $2 \leq t \leq 4$, where $\overline{R} = R/(x_1, x_2) : I^\infty$, x_1, x_2 and x_3 are general elements in I . One has that $\ell(I) = 3$ and I is j -stretched. Since $\dim(R/I) = 1$, the property AN_1^- is automatically satisfied. Moreover, the second ideal has the G_3 condition because I is generically a complete intersection.

Example 2.6. Let $I = (a^2b^2, a^2c^2, abc^2, b^2c^2, a^2bc) \subseteq R = k[a, b, c]$. Then I is a Cohen-Macaulay ideal that is generated in a single degree with AN_1^- . The equality $\lambda(\overline{I^2}/x_3\overline{I} + \overline{I^3}) = 1$, where $\overline{R} = R/(x_1, x_2) : I^\infty$, x_1, x_2 and x_3 are general elements in I , implies that $\ell(I) = 3$ and I is j -stretched.

3. STRUCTURE OF j -STRETCHED IDEALS

In this section we introduce techniques to study the structure of j -stretched ideals. These technical results will be employed in the next section to prove our main theorems. We start with the proof of the Specialization Lemma (Lemma 3.1). To state it, we need to recall the notion of specialization of modules, as introduced by Nhi and Trung [16].

Let $S = R[\underline{z}]$, where $\underline{z} = z_1, \dots, z_t$ are variables over the Noetherian local ring (R, \mathfrak{m}, k) (recall k is infinite and $d = \dim R$). Let M' be a finite S -module. Let $\phi : S^f \rightarrow S^g \rightarrow 0$ be a finite free presentation of M' and let $A = (a_{ij}[\underline{z}])$ be a matrix representation of ϕ . For any vector $\underline{\alpha} = (\alpha_1, \dots, \alpha_t) \in R^t$, let $A_{\underline{\alpha}} := (a_{ij}[\underline{\alpha}])$ and $\phi_{\underline{\alpha}} : R^f \rightarrow R^g \rightarrow 0$ be the corresponding map defined by $A_{\underline{\alpha}}$. One says that $\phi_{\underline{\alpha}}$ is a *specialization* of ϕ . A *specialization* of M' is defined to be $M'_{\underline{\alpha}} := \text{Coker}(\phi_{\underline{\alpha}})$. By [16], $M'_{\underline{\alpha}}$ does not depend on (up to isomorphisms) the choice of ϕ and A . The vector $\underline{\alpha} \in R^t$ is said to be *general* (equivalently, the specialization $M'_{\underline{\alpha}}$ is *general*) if the image $\overline{\underline{\alpha}} = (\overline{\alpha}_1, \dots, \overline{\alpha}_t) \in U$, where U is some Zariski dense open subset of k^t .

Lemma 3.1. [*Specialization Lemma*] *Let S be as above. Let M be a finite R -module and $M' = M \otimes_R S$. Let $N' \subseteq M'$ be a submodule such that $\lambda_{S_{\mathfrak{m}S}}(M'_{\mathfrak{m}S}/N'_{\mathfrak{m}S}) = \delta \in \mathbb{N}_0$. Then*

- (a) *For a general vector $\underline{\alpha} \in R^t$, one has that $\lambda_R(M/N'_{\underline{\alpha}}) = \delta$.*
- (b) *Assume R is equicharacteristic and fix any vector $\underline{\alpha}_0 \in R^t$. Then for a general vector $\underline{\alpha} \in R^t$, one has that $\delta = \lambda_R(M/N'_{\underline{\alpha}}) \leq \lambda_R(M/N'_{\underline{\alpha}_0})$.*

Proof. We may pass to the \mathfrak{m} -adic completion of R to assume that $R \cong A/H$, where A is a regular local ring. We may also replace R by A to assume that R is a regular local ring, and therefore M' is a finite module of a polynomial ring over a regular local ring. We use induction on δ to prove part (a). Notice that this statement holds if $\delta \leq 1$. Indeed, if $\delta = 0$, i.e., $\lambda_{S_{\mathfrak{m}S}}(M'_{\mathfrak{m}S}/N'_{\mathfrak{m}S}) = 0$, then there exists a polynomial $f \in S \setminus \mathfrak{m}S$ such that $fM' \subseteq N'$. Let \bar{f} be the image of f in $k[z_1, \dots, z_t]$ and notice that $\bar{f} \neq 0$. Thus $U = D(\bar{f})$ is a Zariski dense open subset of k^t . If $(\underline{\alpha}) \in U$ then $f(\underline{\alpha})$ is a unit in R . Thus $f(\underline{\alpha})M = (fM')_{\underline{\alpha}} \subseteq N'_{\underline{\alpha}}$ implies $M \subseteq N'_{\underline{\alpha}}$.

We now consider the case $\delta = 1$, i.e., $M'_{\mathfrak{m}S}/N'_{\mathfrak{m}S} \cong k(\underline{z})$. In this case there exists $\xi \in M \setminus N'$ such that $M'_{\mathfrak{m}S} = \xi S_{\mathfrak{m}S} + N'_{\mathfrak{m}S}$ and $\xi \mathfrak{m}S_{\mathfrak{m}S} \subseteq N'_{\mathfrak{m}S}$. Hence, there exists a polynomial $f \in S \setminus \mathfrak{m}S$ such that $fM' \subseteq \xi S + N'$ and $f\xi \mathfrak{m}S \subseteq N'$. Again $0 \neq \bar{f}$ is the image of f in $k[z_1, \dots, z_t]$ and $U = D(\bar{f})$ is a Zariski dense open subset of k^t . If $(\underline{\alpha})$ lies in U then $f(\underline{\alpha})$ is a unit in R , and therefore we have $M = M'_{\underline{\alpha}} = \xi R + N'_{\underline{\alpha}}$ and $\xi \mathfrak{m} \subseteq N'_{\underline{\alpha}}$.

Set $I = \text{Ann}_S(\xi S + N')/N'$ and notice that $I_{\mathfrak{m}S} = \mathfrak{m}S_{\mathfrak{m}S}$. By the properties of general specialization (see [16], which still hold because we are over a regular local ring), there exists a Zariski dense open subset V of k^t such that for every $\underline{\alpha} \in V$ one has

$$M/N'_{\underline{\alpha}} = M'_{\underline{\alpha}}/N'_{\underline{\alpha}} = \xi R + N'_{\underline{\alpha}}/N'_{\underline{\alpha}} \cong R/\text{Ann}_R[(\xi R + N'_{\underline{\alpha}})/N'_{\underline{\alpha}}] \cong [S/I]_{\underline{\alpha}}.$$

Therefore, we have

$$\begin{aligned} \lambda_R(M/N'_{\underline{\alpha}}) &= \lambda_R([S/I]_{\underline{\alpha}}) &&= \lambda_R(S/I \otimes_S S/(\underline{z} - \underline{\alpha})) \\ &= \lambda_R(S/I \otimes_S S_{\mathfrak{m}S}/(\underline{z} - \underline{\alpha})S_{\mathfrak{m}S}) &&= \lambda_R(S_{\mathfrak{m}S}/I_{\mathfrak{m}S} \otimes_{S_{\mathfrak{m}S}} S_{\mathfrak{m}S}/(\underline{z} - \underline{\alpha})S_{\mathfrak{m}S}) \\ &= \lambda_R(S_{\mathfrak{m}S}/\mathfrak{m}S_{\mathfrak{m}S} \otimes_{S_{\mathfrak{m}S}} S_{\mathfrak{m}S}/(\underline{z} - \underline{\alpha})S_{\mathfrak{m}S}) &&= \lambda_R(S/\mathfrak{m}S \otimes_S S/(\underline{z} - \underline{\alpha})S \otimes_S S_{\mathfrak{m}S}) \\ &= \lambda_R(k) &&= 1. \end{aligned}$$

We may then assume $\delta > 1$ and assertion (a) holds for $\delta - 1$. For every element $x \in M$, write $x' = x \otimes 1$ in $M \otimes_R S = M'$. Notice that if every element $x \in M$ has the property that $\frac{x'}{1} \in N'_{\mathfrak{m}S}$, then $M'_{\mathfrak{m}S} = N'_{\mathfrak{m}S}$, showing that $\delta = 0$, which is a contradiction.

Hence, there exists an element $x_0 \in M$ with $\frac{x'_0}{1} \notin N'_{\mathfrak{m}S}$. We claim that we can choose $x \in M$ with the property that $\frac{x'}{1} \in M'_{\mathfrak{m}S} \setminus N'_{\mathfrak{m}S}$ and $x'\mathfrak{m}S_{\mathfrak{m}S} \subseteq N'_{\mathfrak{m}S}$. Let $\gamma = \max\{n \in \mathbb{N} \mid x'_0 \mathfrak{m}^n S_{\mathfrak{m}S} \not\subseteq N'_{\mathfrak{m}S}\}$. Notice that $0 \leq \gamma \leq \delta - 1$. If for every element $a \in \mathfrak{m}^\gamma R$ we have $\frac{(ax_0)'}{1} \in N'_{\mathfrak{m}S}$, then $x'_0 \mathfrak{m}^\gamma S_{\mathfrak{m}S} \subseteq N'_{\mathfrak{m}S}$, that is a contradiction. Hence, there exists an element $a \in \mathfrak{m}^\gamma R$ with $\frac{(ax_0)'}{1} \in M'_{\mathfrak{m}S} \setminus N'_{\mathfrak{m}S}$, but $(ax_0)'\mathfrak{m}S_{\mathfrak{m}S} \subseteq N'_{\mathfrak{m}S}$. Since $a \in \mathfrak{m}^\gamma$ and $x_0 \in M$, it follows that $x = ax_0 \in M$ and has the property that $\frac{x'}{1} \in M'_{\mathfrak{m}S} \setminus N'_{\mathfrak{m}S}$ and $x'\mathfrak{m}S_{\mathfrak{m}S} \subseteq N'_{\mathfrak{m}S}$. Now, set $N'_{\delta-1} = N' + x'S \subseteq M'$. By construction, we have

$$\lambda_{S_{\mathfrak{m}S}}((N'_{\delta-1}/N')_{\mathfrak{m}S}) = 1.$$

Since $(N'_{\delta-1}/N')_{\mathfrak{m}S} \cong (x'S/(x'S \cap N'))_{\mathfrak{m}S}$ and $x'S = xR \otimes_R S$ for an R -submodule $xR \subseteq M$, by the case of $\delta = 1$, we have $\lambda_R((N'_{\delta-1}/N')_{\alpha}) = 1$ for a general α . Also, by the above, we obtain $\lambda_{S_{\mathfrak{m}S}}((M'/N'_{\delta-1})_{\mathfrak{m}S}) = \lambda_{S_{\mathfrak{m}S}}((M'/N')_{\mathfrak{m}S}) - \lambda_{S_{\mathfrak{m}S}}((N'_{\delta-1}/N')_{\mathfrak{m}S}) = \delta - 1$. Hence, by induction hypothesis, for a general vector α , one has

$$\lambda_R((M'/N'_{\delta-1})_{\alpha}) = \delta - 1,$$

proving that, for a general vector α , we have

$$\lambda_R(M/N'_{\underline{\alpha}}) = \lambda_R((M'/N')_{\alpha}) = \lambda_R((M'/N'_{\delta-1})_{\alpha}) + \lambda_R((N'_{\delta-1}/N')_{\alpha}) = \delta - 1 + 1 = \delta.$$

To prove part (b), first notice that if $\lambda_R(M/N'_{\underline{\alpha}_0}) = \infty$ then there is nothing to prove. Hence we may assume $\lambda_R(M/N'_{\underline{\alpha}_0}) < \infty$.

Since $\lambda_{S_{\mathfrak{m}S}}(M'_{\mathfrak{m}S}/N'_{\mathfrak{m}S}) < \infty$ and $\lambda_R(M/N'_{\underline{\alpha}_0}) < \infty$, there exists a positive integer t_0 such that $\mathfrak{m}^{t_0} M'_{\mathfrak{m}S} \subseteq N'_{\mathfrak{m}S}$ and $\mathfrak{m}^{t_0} M \subseteq N'_{\underline{\alpha}_0}$. Thus, there exists an element $f \in S \setminus \mathfrak{m}S$ such that $f\mathfrak{m}^{t_0} M' \subseteq N'$. Then for every $\underline{\alpha} \in D(\overline{f})$, one has that $\mathfrak{m}^{t_0} M \subseteq N'_{\underline{\alpha}}$.

Since R is equicharacteristic, R contains its residue field k . Theretofore for every $\underline{\alpha}$ in $U_1 = D(\overline{f}) \cup \{\underline{\alpha}_0\}$, we have the following isomorphisms of $S/\mathfrak{m}^{t_0}S$ -modules:

$$\begin{aligned} M'/N' \otimes_S S/\mathfrak{m}^{t_0}S \otimes_{k[\underline{z}]/(\underline{z}-\underline{\alpha})k[\underline{z}]} k[\underline{z}]/(\underline{z}-\underline{\alpha})k[\underline{z}] &\cong M'/N' \otimes_S S/\mathfrak{m}^{t_0}S \otimes_{k[\underline{z}]} k[\underline{z}]/(\underline{z}-\underline{\alpha}) \\ &\cong S/\mathfrak{m}^{t_0}S \otimes_S (M'/N' \otimes_{k[\underline{z}]} k[\underline{z}]/(\underline{z}-\underline{\alpha})) \\ &\cong S/\mathfrak{m}^{t_0}S \otimes_S M/N'_{\underline{\alpha}} \\ &\cong M/N'_{\underline{\alpha}} \end{aligned}$$

where the last isomorphism follows because $M/N'_{\underline{\alpha}}$ is an $S/\mathfrak{m}^{t_0}S$ module. Notice that $M'/N' \otimes_S S/\mathfrak{m}^{t_0}S$ is a finite $k[\underline{z}]$ -module (M'/N' is a finite S -module and $S/\mathfrak{m}^{t_0}S \cong R/\mathfrak{m}^{t_0} \otimes_k k[\underline{z}]$ is a finite $k[\underline{z}]$ -module). Hence for every $\underline{\alpha} \in U_1$ we have

$$\begin{aligned} &\mu_{k[\underline{z}]/(\underline{z}-\underline{\alpha})k[\underline{z}]}((M'/N' \otimes_S S/\mathfrak{m}^{t_0}S)_{(\underline{z}-\underline{\alpha})k[\underline{z}]}) \\ &= \lambda_{k[\underline{z}]/(\underline{z}-\underline{\alpha})k[\underline{z}]}(M'/N' \otimes_S S/\mathfrak{m}^{t_0}S \otimes_{k[\underline{z}]/(\underline{z}-\underline{\alpha})k[\underline{z}]} k[\underline{z}]/(\underline{z}-\underline{\alpha})) \\ &= \lambda_R(M/N'_{\underline{\alpha}}), \end{aligned}$$

where the last equality follows by the above isomorphisms, and the first equality holds by Nakayama's Lemma (that can be applied because $(M'/N' \otimes_S S/\mathfrak{m}^{t_0}S)_{(\underline{z}-\underline{\alpha})k[\underline{z}]}$ is finite over $k[\underline{z}]/(\underline{z}-\underline{\alpha})k[\underline{z}]$ by the above). Set $q = \lambda_R(M/N'_{\underline{\alpha}_0})$, then one has the following Zariski open subset of k^t

$$U_2 = \{\underline{\alpha} \in k^t \mid \mu_{k[\underline{z}]/(\underline{z}-\underline{\alpha})k[\underline{z}]}((M'/N' \otimes_S S/\mathfrak{m}^{t_0}S)_{(\underline{z}-\underline{\alpha})k[\underline{z}]}) \leq q\} = k^t \setminus V(\text{Fitt}_q(M'/N' \otimes_S S/\mathfrak{m}^{t_0}S)).$$

Notice U_2 is dense because $\underline{\alpha}_0 \in U_2$. Finally for any $\underline{\alpha} \in U = U_1 \cap U_2$ which is again a Zariski dense open subset of k^t , we have

$$\lambda_R(M/N'_{\underline{\alpha}}) = \mu_{k[\underline{z}]/(\underline{z}-\underline{\alpha})k[\underline{z}]}((M'/N' \otimes_S S/\mathfrak{m}^{t_0}S)_{(\underline{z}-\underline{\alpha})k[\underline{z}]}) \leq q = \lambda_R(M/N'_{\underline{\alpha}_0}).$$

□

Lemma 3.1 greatly enhances our ability to study arbitrary ideals and modules. Indeed we are going to apply it to study the index of nilpotency of any ideal. For this purpose, we recall that the *index of nilpotency* of an R -ideal I with respect to a reduction J is defined to be the integer

$$s_J(I) = \min\{n \mid I^{n+1} \subseteq J\}.$$

In Proposition [5, 5.3.3], Fouli proved that the index of nilpotency of \mathfrak{m} -primary ideals over an equicharacteristic Cohen-Macaulay local ring does not depend on the general minimal reduction, and general minimal reductions achieve the largest possible index of nilpotency. We generalize this result to non \mathfrak{m} -primary ideals using Lemma 3.1 as a crucial ingredient (see the following proposition).

Proposition 3.2. *Assume R is Cohen-Macaulay. Let I be an R -ideal which has $\ell(I) = d$ and the G_d condition. Let J be a general minimal reduction of I . Then $s_J(I)$ does not depend on a choice of J . Furthermore, assume R is equicharacteristic, and either I is \mathfrak{m} -primary or I satisfies AN_{d-2}^- , $\text{depth}(R/I) \geq 1$, and $J \cap I^2 = JI$. Let H be any fixed minimal reduction of I . Then $s_H(I) \leq s_J(I)$.*

Proof. First by the following exact sequence

$$0 \rightarrow J + I^{s+1}/J \rightarrow I/J \rightarrow I/J + I^{s+1} \rightarrow 0,$$

one has that $\lambda(J + I^{s+1}/J) = \lambda(I/J) - \lambda(I/J + I^{s+1})$. By Lemma 3.1 (see also Proposition 5.1 in Section 5), the lengths $\lambda(I/J)$ and $\lambda(I/J + I^{s+1})$ do not depend on J . Therefore $\lambda(J + I^{s+1}/J)$ and thus $s_J(I)$ do not depend on a choice of the general minimal reduction J .

Next assume R is equicharacteristic. The case where I is \mathfrak{m} -primary has been proved by Proposition [5, 5.3.3]. So we may assume that I satisfies AN_{d-2}^- , $\text{depth}(R/I) \geq 1$, and $J \cap I^2 = JI$. Write $J = (x_1, \dots, x_d)$, where x_1, \dots, x_d are general elements in I . Set $J_{d-1} = (x_1, \dots, x_{d-1})$. By [18, Lemma 3.2], $J_{d-1} : I$ is a geometric $d - 1$ -residual intersection of I , $(J_{d-1} : I) \cap I = J_{d-1}$, and $(J_{d-1} : I) \cap I^2 = J_{d-1} \cap I^2 = J_{d-1}I$. Since I satisfies AN_{d-2}^- , one has that $J_{d-1} : I^\infty = J_{d-1} : I$. Let $\overline{R} = R/J_{d-1} : I$ and use $\overline{}$ to denote images in the quotient ring \overline{R} . By the above and the proof of [18, Proposition 2.1], one has

$$\begin{aligned} j(I) &= e(\overline{I}, \overline{R}) = \lambda(\overline{I}/x_d \overline{I}) = \lambda(I/(J_{d-1} : I) \cap I + x_d I) \\ &= \lambda(I/J_{d-1} + x_d I) = \lambda(I/J_{d-1} + I^2) + \lambda(J_{d-1} + I^2/J_{d-1} + x_d I) \\ &= \lambda(I/J_{d-1} + I^2) + \lambda(I^2/J_{d-1} \cap I^2 + x_d I) = \lambda(I/J_{d-1} + I^2) + \lambda(I^2/JI). \end{aligned}$$

Now consider the minimal reduction H of I . Since $H : I = R$, one has that $\text{ht } H : I = \infty$. By [30, Lemma 1.4], one can choose elements y_1, \dots, y_d in H such that $H = (y_1, \dots, y_d)$ and $H_{d-1} : I$, where $H_{d-1} = (y_1, \dots, y_{d-1})$, is a geometric $d - 1$ -residual intersection of I . Since I satisfies AN_{d-2}^- and $\text{depth}(R/I) \geq 1$, one has that $H_{d-1} : I^\infty = H_{d-1} : I$, $(H_{d-1} : I) \cap I = H_{d-1}$, and $(H_{d-1} : I) \cap I^2 = H_{d-1} \cap I^2 = H_{d-1}I$ (see [14, Lemmas 2.3 and 2.4]). Moreover by avoiding finitely many more prime ideals, one can also assume that y_1, \dots, y_d form a super-reduction for I (in the sense of Achilles and Manaresi [3]). Therefore we can use y_1, \dots, y_d to compute the j -multiplicity $j(I)$ [3, 3.8]. Let $R' = R/H_{d-1} : I$ and use $'$ to denote images in the quotient ring R' . By the same argument as above, one has

$$j(I) = e(I', R') = \lambda(I'/x_d I') = \lambda(I/H_{d-1} + I^2) + \lambda(I^2/HI).$$

By Lemma 3.1 (see also Proposition 5.1 in Section 5), one has that $\lambda(I/J_{d-1} + I^2) \leq \lambda(I/H_{d-1} + I^2)$ and $\lambda(I^2/JI) \leq \lambda(I^2/HI)$, thus, $\lambda(I^2/JI) = \lambda(I^2/HI)$. From the exact sequences

$$\begin{aligned} 0 \rightarrow HI + I^{s+1}/HI &\rightarrow I^2/HI \rightarrow I^2/HI + I^{s+1} \rightarrow 0, \\ 0 \rightarrow JI + I^{s+1}/JI &\rightarrow I^2/JI \rightarrow I^2/JI + I^{s+1} \rightarrow 0, \end{aligned}$$

one deduces $\lambda(HI + I^{s+1}/HI) \leq \lambda(JI + I^{s+1}/JI)$. Let $s = s_J(I)$. If $s = 0$, the statement follows. Otherwise, one has $I^{s+1} \subseteq J \cap I^2 = JI$, whence $\lambda(JI + I^{s+1}/JI) = 0$. Therefore, one obtains the equality $\lambda(HI + I^{s+1}/HI) = 0$, which, in turn, implies $I^{s+1} \subseteq HI \subseteq H$. \square

Let I be an R -ideal which has $\ell(I) = d$ and the G_d condition. We then define the *index of nilpotency* of I as $s(I) = s_J(I)$, where J is a general minimal reduction of I . This number is well-defined by Proposition 3.2. We set two typical settings for our next results.

Setting 3.3. *Let R be Cohen-Macaulay and I an R -ideal. Assume either*

- (1) $\ell(I) = d$ and I satisfies G_d condition, AN_{d-2}^- , and $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$.
- (2) or I is \mathfrak{m} -primary and $J_{d-1} \cap I^2 \subseteq JI$, where $J = (x_1, \dots, x_d)$ is a general minimal reduction of I and $J_{d-1} = (x_1, \dots, x_{d-1})$.

The following lemmas generalize to j -stretched ideals the corresponding results of stretched \mathfrak{m} -primary ideals proved in [23]. Since the associated graded rings of ideals having minimal j -multiplicity are known to be Cohen-Macaulay [18, Theorem 3.9], we can harmlessly assume that I does not have minimal j -multiplicity.

Lemma 3.4. *Let R and I be as in Setting 3.3. Let I be j -stretched, not having minimal j -multiplicity. Then*

- (a) $j(I) \geq \lambda(\overline{R}/\overline{I}) + h + 1$, where $h = \lambda(\overline{I}/\overline{I}^2) - \lambda(\overline{R}/\overline{I})$, $\overline{R} = R/J_{d-1} : I^\infty$.
- (b) For every $n \geq 1$, we have $I^{n+1} = JI^n + (a^n b)$, where $a, b \in I$ and $a, b \notin J$.
- (c) For every $n \geq 1$, we have $a^n b \mathfrak{m} \subseteq I^{n+2} + JI^n$.
- (d) $I = (b) + (J : a) \cap I$.

Proof. (a) By [18, Proposition 2.1], we have $j(I) = e(I, \overline{R}) = \lambda(\overline{I}/\overline{I}^2) + \lambda(\overline{I}^2/x_d \overline{I})$. By definition of h , this equals $\lambda(\overline{R}/\overline{I}) + h + \lambda(\overline{I}^2/x_d \overline{I})$. Hence, to finish the proof of (a), we have to show that $\lambda(\overline{I}^2/x_d \overline{I}) \geq 1$, which holds because the length is 0 if and only if I has minimal j -multiplicity.

To prove assertion (b), we first show $\lambda(I^2/JI + I^3) = 1$. Since I is j -stretched and does not have minimal j -multiplicity, one has that $\lambda(\overline{I}^2/x_d \overline{I} + \overline{I}^3) = 1$ (otherwise $\overline{I}^2 = x_d \overline{I} + \overline{I}^3$ which implies $\overline{I}^2 = x_d \overline{I}$ by Nakayama's Lemma). Notice $\lambda(\overline{I}^2/x_d \overline{I} + \overline{I}^3) = \lambda[I^2/((J_{d-1} : I^\infty) \cap I^2 + x_d I + I^3)] = 1$. We need to show $(J_{d-1} : I^\infty) \cap I^2 = J_{d-1} I$, which immediately yields $\lambda(I^2/JI + I^3) = 1$. The case where I is not \mathfrak{m} -primary has been proved by [18, Lemma 3.2]. So assume I is \mathfrak{m} -primary. Since $J_{d-1} \cap I^2 \subseteq JI$ and x_d is a non zero divisor on R/J_{d-1} , one has

$$(J_{d-1} : I) \cap I^2 = J_{d-1} \cap I^2 = J_{d-1} \cap JI = J_{d-1} \cap (J_{d-1} I + x_d I) = J_{d-1} I + J_{d-1} \cap x_d I$$

$$= J_{d-1}I + x_d(J_{d-1} : I \ x_d) = J_{d-1}I + x_d J_{d-1} = J_{d-1}I.$$

Now we use induction on n to prove assertion (b). The length $\lambda(I^2/JI + I^3) = 1$ implies that $I^2 = JI + I^3 + (ab)$ for some $a, b \in I \setminus J$. By Nakayama's Lemma, $I^2 = JI + (ab)$, proving the statement in the case $n = 1$. For any $n \geq 1$, assume $I^{n+1} = JI^n + (a^n b)$. We need to show that $I^{n+2} = JI^{n+1} + (a^{n+1}b)$. This holds since $I^{n+2} = I(JI^n + (a^n b)) = JI^{n+1} + a^n(bI) \subseteq JI^{n+1} + a^n(JI + (ab)) = JI^{n+1} + (a^{n+1}b)$.

The proofs of (c) and (d) are similar to the corresponding statements in [23, Lemma 2.4]. We write them for the sake of completeness. Assertion (c) can be proved by induction on $n \geq 1$. The case $n = 1$ follows from the facts that $\lambda(I^2/JI + I^3) = 1$ and $I^2 = JI + (ab)$. Now assume $n \geq 1$ and $(a^n b)\mathfrak{m} \subseteq I^{n+2} + JI^n$. Then one has

$$(a^{n+1}b)\mathfrak{m} = a[(a^n b)\mathfrak{m}] \subseteq a[I^{n+2} + JI^n] \subseteq I^{n+3} + JI^{n+1}.$$

(d) Since $aI \subseteq I^2 = JI + (ab)$, we have $I \subseteq (JI + (ab)) : a$. The easily checked equality $(JI + (ab)) : a = (JI : a) + (b)$ now implies

$$I \subseteq [(JI : a) + (b)] \cap I \subseteq (b) + (J : a) \cap I \subseteq I.$$

□

Let I be an R -ideal which has maximal analytic spread $\ell(I) = d$ and the G_d condition. Let $J = (x_1, \dots, x_d)$ be a general minimal reduction of I . Recall $\overline{R} = R/J_{d-1} : I^\infty$, where $J_{d-1} = (x_1, \dots, x_{d-1})$. We set $\nu_n = \lambda(I^{n+1}/JI^n)$ and $\overline{\nu}_n = \lambda(\overline{I^{n+1}}/\overline{JI^n})$ for every $n \geq 0$, which are well-defined by Lemma 3.1.

Lemma 3.5. *Let R and I be as in Setting 3.3. If I is j -stretched, then*

- (a) $\nu_n \leq \nu_{n-1}$ for every $n \geq 2$.
- (b) $\overline{\nu}_1 = \nu_1$ and $\overline{\nu}_n \leq \nu_n$ for every $n \geq 2$.

Proof. (a) If I has minimal j -multiplicity then $r(I) \leq 1$ (see [18, Theorem 3.3]). Hence $\nu_n = \lambda(I^{n+1}/JI^n) = 0$ for every $n \geq 1$. Assume I does not have minimal j -multiplicity. Let a be the same as in Lemma 3.4 (b). Then for every $n \geq 2$, we have the following epimorphism

$$I^n/JI^{n-1} \xrightarrow{\cdot a} I^{n+1}/JI^n \longrightarrow 0.$$

Hence $\nu_n = \lambda(I^{n+1}/JI^n) \leq \nu_{n-1} = \lambda(I^n/JI^{n-1})$.

(b) For any n , we have the natural epimorphism

$$I^{n+1}/JI^n \longrightarrow \overline{I^{n+1}}/\overline{JI^n} \longrightarrow 0,$$

inducing the inequality $\overline{\nu}_n \leq \nu_n$. Furthermore, one has

$$\begin{aligned} \overline{\nu}_1 &= \lambda(\overline{I^2}/\overline{JI}) = \lambda(I^2/(J_{d-1} : I^\infty) \cap I^2 + JI) \\ &= \lambda(I^2/JI) = \nu_1, \end{aligned}$$

where the third equality follows from the fact $(J_{d-1} : I^\infty) \cap I^2 = J_{d-1}I$ (see [18, Lemma 3.2] and the proof of Lemma 3.4).

□

Now consider the Hilbert function $H_{I, \overline{R}}(n) = \lambda(\overline{I}^n / \overline{I}^{n+1})$, which does not depend on J (see [19]). In particular, it is well-defined the integer $h = \lambda(\overline{I} / \overline{I}^2) - \lambda(\overline{R} / \overline{I})$, which is dubbed the *embedding codimension* of I . Moreover, one has that (see [18] and [23])

$$j(I) = e(I, \overline{R}) = \lambda(\overline{R} / \overline{I}) + h + K - 1, \text{ where } K - 1 = \lambda(\overline{I}^2 / x_d \overline{I}).$$

The following corollary shows that if I is j -stretched, then K is the index of nilpotency $s(I)$.

Corollary 3.6. *Let R and I be as in Setting 3.3. If I is j -stretched, then*

$$\nu_1 = K - 1, \quad I^K \not\subseteq J, \quad I^{K+1} \subseteq J.$$

Proof. By the proof of Lemma 3.5 (b), we have $K - 1 = \lambda(\overline{I}^2 / x_d \overline{I}) = \lambda(I^2 / JI) = \nu_1$. Since I is j -stretched, one has that

$$P_{I, \overline{R}/x_d \overline{R}} = \lambda(\overline{R} / \overline{I}) + hz + z^2 + \dots + z^{j(I)-h+1-\lambda(\overline{R}/\overline{I})}.$$

Therefore K is the least positive integer with

$$I^{K+1} \subseteq [(J_{d-1} : I^\infty) \cap I^{K+1}] + I^{K+2} + J \subseteq I^{K+2} + J.$$

By Nakayama's Lemma, K is the least positive integer with $I^{K+1} \subseteq J$. □

The next result is the last ingredient that we need to characterize the Cohen-Macaulayness of $\text{gr}_I(R)$ when I is j -stretched. It shows that the inclusion $I^{K+1} \subseteq JI^n$ is equivalent to certain Valabrega-Valla equalities for small powers of I . More precisely,

Proposition 3.7. *Let R and I be as in Setting 3.3. If I is j -stretched with index of nilpotency K , then for any $0 \leq n \leq K$, one has:*

- (a) $J \cap I^{n+1} = JI^n + (a^K b)$, where a and b are as in Lemma 3.4 (b).
- (b) $I^{K+1} \subseteq JI^n$ if and only if $J \cap I^{t+1} = JI^t$ for every $t \leq n$.

Proof. (a) We use descending induction on $n \leq K$. When $n = K$, by Lemma 3.4 (b), $J \cap I^{K+1} = I^{K+1} = JI^K + (a^K b)$. Now assume $J \cap I^{n+1} = JI^n + (a^K b)$ and prove $J \cap I^n = JI^{n-1} + (a^K b)$. One inclusion $JI^{n-1} + (a^K b) \subseteq J \cap I^n$ is clear. We prove $J \cap I^n \subseteq JI^{n-1} + (a^K b)$. By Lemma 3.4 (b), $J \cap I^n = J \cap (JI^{n-1} + (a^{n-1} b)) = JI^{n-1} + (a^{n-1} b) \cap J$. Since $I^n \not\subseteq J$ by Corollary 3.6, one has

$$\begin{aligned} (a^{n-1} b) \cap J &\subseteq a^{n-1} b \mathfrak{m} \cap J \subseteq (I^{n+1} + JI^{n-1}) \cap J \\ &= (I^{n+1} \cap J) + JI^{n-1} = JI^n + (a^K b) + JI^{n-1} = JI^{n-1} + (a^K b). \end{aligned}$$

The proof of assertion (b) is similar to the one of [23, Lemma 2.5 (ii)]. □

4. COHEN-MACAULAYNESS AND ALMOST COHEN-MACAULAYNESS OF $\text{gr}_I(R)$

In this section we study the depth of the associated graded rings $\text{gr}_I(R)$ of j -stretched ideals. In Theorem 4.1, we prove that $\text{gr}_I(R)$ is Cohen-Macaulay if and only if the reduction number and the index of nilpotency of the ideal I are equal. We also prove Sally's conjecture for j -stretched ideals, providing a sufficient condition for $\text{gr}_I(R)$ to be almost Cohen-Macaulay (see Theorem 4.6). Our work combines the approaches of Rossi-Valla and Polini-Xie and generalizes widely the main results of [26], [23] and [18].

Theorem 4.1. *Let R and I be as in Setting 3.3. Let I be j -stretched with the index of nilpotency K . Then the following statements are equivalent:*

- (a) $\text{gr}_I(R)$ is Cohen-Macaulay.
- (b) $r(I) = K$.

Furthermore, if R is equicharacteristic, then (a) and (b) are also equivalent to

- (c) $I^{K+1} = HI^K$ for some minimal reduction H of I .

Proof. We first prove the following two claims.

Claim 1. *The equalities $J \cap I^{n+1} = JI^n$ hold for every $n \geq 0$ if and only if $I^{K+1} = JI^K$.*

The forward direction is straightforward since $I^{K+1} \subseteq J$. Conversely, if $I^{K+1} = JI^K$, then by Proposition 3.7 (b), one has $J \cap I^{n+1} = JI^n$ for every $0 \leq n \leq K$. If $n \geq K$, one has $I^{n+1} = I^{n-K}I^{K+1} = I^{n-K}JI^K = JI^n$ and then obtains $J \cap I^{n+1} = JI^n$.

Claim 2. *Write $g = \text{grade } I$. Let x_1^*, \dots, x_d^* be the initial forms of x_1, \dots, x_d in $\text{gr}_I(R)$. If $I^{K+1} = JI^K$, then x_1^*, \dots, x_g^* form a regular sequence on $\text{gr}_I(R)$.*

Since x_1, \dots, x_g are general elements in I and $g = \text{grade } I$, then x_1, \dots, x_g form a regular sequence on R . By Valabrega-Valla criterion (see [31, Proposition 2.6] or [24, Theorem 1.1]), we only need to show $(x_1, \dots, x_g) \cap I^n = (x_1, \dots, x_g)I^{n-1}$ for every $n \geq 1$. The case where I is \mathfrak{m} -primary follows from [23, Theorem 2.6], hence we may assume $\dim R/I > 0$. We use induction on n to prove $(x_1, \dots, x_i) \cap I^n = (x_1, \dots, x_i)I^{n-1}$ for every $n \geq 1$ and $0 \leq i \leq d$. This is clear if $n = 1$. We then assume $n \geq 2$ and the equality holds for $n - 1$. Now, we use descending induction on $i \leq d$. Since I is j -stretched with $I^{K+1} = JI^K$, then, by Claim 1, $J \cap I^n = JI^{n-1}$, which proves the case $i = d$. Now assume $i < d$ and, by induction, that $(x_1, \dots, x_{i+1}) \cap I^n = (x_1, \dots, x_{i+1})I^{n-1}$. Then

$$\begin{aligned}
& (x_1, \dots, x_i)I^{n-1} \subseteq (x_1, \dots, x_i) \cap I^n \\
= & (x_1, \dots, x_i) \cap (x_1, \dots, x_{i+1})I^{n-1} && \text{by induction on } i \\
= & (x_1, \dots, x_i) \cap ((x_1, \dots, x_i)I^{j-1} + x_{i+1}I^{n-1}) \\
= & (x_1, \dots, x_i)I^{n-1} + (x_1, \dots, x_i) \cap x_{i+1}I^{n-1} \\
= & (x_1, \dots, x_i)I^{n-1} + x_{i+1}[(x_1, \dots, x_i) : x_{i+1}] \cap I^{n-1} \\
= & (x_1, \dots, x_i)I^{n-1} + x_{i+1}[(x_1, \dots, x_i) \cap I^{n-1}] && \text{by [18, Lemma 3.2]} \\
= & (x_1, \dots, x_i)I^{n-1} + x_{i+1}(x_1, \dots, x_i)I^{n-2} && \text{by induction on } n \\
\subseteq & (x_1, \dots, x_i)I^{n-1}
\end{aligned}$$

which yields the desired equality.

We are now ready to prove the theorem.

(a) \iff (b). The proof is similar to [18, Theorem 3.8]. Set $\delta(I) = d - g$. We prove the equivalence of (a) and (b) by induction on $\delta(I)$. If $\delta(I) = 0$, the assertion follows because we proved in Claim 2 that x_1^*, \dots, x_g^* form a regular sequence on $\text{gr}_I(R)$. Thus we may assume that $\delta(I) \geq 1$ and the theorem holds for smaller values of $\delta(I)$. In particular, $d \geq g + 1$. Since in both cases x_1^*, \dots, x_g^* form a regular sequence on $\text{gr}_I(R)$, we may factor out x_1, \dots, x_g to assume $g = 0$. Now $d = \delta(I) \geq 1$. Set $R' = R/H_0$, where $H_0 = 0 : I$, and use $'$ to denote images in R' . By [18, Lemma 3.2], one has $I \cap H_0 = 0$, R' is Cohen-Macaulay with $\dim R' = d$, $\text{grade}(I') \geq 1$, $\ell(I') = d$, I' still satisfies G_d and AN_{d-2}^- on R' and $\text{depth}(R'/I') \geq \min\{\dim R'/I', 1\}$. By the definition of j -stretchedness, one has that I' is j -stretched in R' with $K = s(I')$. Since $\delta(I') = d - \text{grade}(I') < d = \delta(I)$, by induction hypothesis, $\text{depth}(\text{gr}_{I'}(R')) \geq d$ if and only if $I'^{K+1} = J'I'^K$. Because $I \cap H_0 = 0$, one has $I'^{K+1}/J'I'^K \cong I^{K+1}/JI^K$ and the following exact sequence

$$(1) \quad 0 \rightarrow H_0 \rightarrow \text{gr}_I(R) \rightarrow \text{gr}_{\overline{I}}(\overline{R}) \rightarrow 0.$$

Notice that $\text{depth}_{\text{m}_G}(R/H_0) = d$. Hence, we have $\text{gr}_I(R)$ is Cohen-Macaulay if and only if $\text{gr}_{I'}(R')$ is Cohen-Macaulay if and only if $I^{K+1} = JI^K$, i.e., $r(I) = K$.

Finally, we assume R is equicharacteristic and prove (b) \iff (c). Clearly (b) implies (c). To prove the converse, notice that, for a general minimal reduction J and a fixed minimal reduction H of I , Lemma 3.1 implies that $\lambda(I^{t+1}/JI^t) \leq \lambda(I^{t+1}/HI^t)$ for $t \geq 0$ (see also the proof of Proposition 5.1 in Section 5). Therefore,

$$K = s(I) \leq r(I) \leq r_H(I).$$

If (c) holds then one has $r_H(I) = K$ which, in turn, yields $K = s(I) = r(I)$. \square

As an immediate application, we recover one of the two main results of Polini-Xie.

Corollary 4.2. ([18, Theorem 3.9]) *Let R be a d -dimensional Cohen-Macaulay local ring and I an R -ideal with $\ell(I) = d$. Assume $\text{depth}(R/I) \geq \min\{\dim(R/I), 1\}$ and I satisfies G_d and AN_{d-2}^- . If I has minimal j -multiplicity then $\text{gr}_I(R)$ is Cohen-Macaulay.*

Proof. If I has minimal j -multiplicity then $r(I) \leq 1$ (see [18, Theorem 3.4]). Hence $K = 1$ and a straightforward application of Theorem 4.1 concludes the proof. \square

In the following, we provide examples of j -stretched ideals which satisfy the assumptions of Theorem 4.1, and therefore their associated graded rings $\text{gr}_I(R)$ are Cohen-Macaulay by our theorem. Notice that the reduction number of the j -stretched ideal I in Example 4.3 could be arbitrarily large, hence, none of the previous criteria in the literature proves the Cohen-Macaulayness of $\text{gr}_I(R)$.

Example 4.3. Fix any $r \geq 1$. Let $R = \mathbb{C}[[x, y, z]]/(x^{r+1}, xz, yz)$ and $I = (x, y)$. We have seen in Example 2.2 that R is a 1-dimensional Cohen-Macaulay local ring and I is a Cohen-Macaulay ideal of height 0 which has $\ell(I) = 1$, G_1 condition, and AN_{-1}^- . The ideal I is also j -stretched with reduction number r (if $r > 2$ then I does not have almost minimal j -multiplicity). By computations, $s(I) = r = r(I)$. Hence by Theorem 4.1, one has that $\text{gr}_I(R)$ is Cohen-Macaulay (indeed, by computations, $\text{gr}_I(R) \cong \mathbb{C}[x, y, z, t, u]/(x, y, zu, t^{r+1}, zt)$).

Example 4.4. Let I be one of the following ideals:

- $I \subseteq R = k[a, b, c]$ is the defining ideal of $n = 6$ generic points of \mathbb{P}^2 .
- $I \subseteq R = k[a, b, c, d]$ is the defining ideal of $n = 4$ or $n = 5$ generic points of \mathbb{P}^3 .
- $I = (a^2, ac, bc, bd, cd) \subseteq R = k[a, b, c, d]$.
- $I = (ab, ac, ad, bc, bd, cd) \subseteq R = k[a, b, c, d]$.
- $I = (a^2, b^2, ad, bd, cd) \subseteq R = k[a, b, c, d]$.
- $I = (a^2, b^2, c^2, ab, bc, cd, de) \subseteq R = k[a, b, c, d, e]$.

Then, I satisfies all the assumptions of Theorem 4.1 and $r(I) = 2 = s(I)$. Therefore, $\text{gr}_I(R)$ is Cohen-Macaulay.

The following theorems (Theorems 4.5 and 4.6) provide a sufficient condition for $\text{gr}_I(R)$ to be almost Cohen-Macaulayness, where I is a j -stretched ideal. They generalize [24, Theorem 4.4], [18, Theorem 4.7] and [18, Theorem 4.10].

Theorem 4.5. Let R be a 2-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be a j -stretched ideal such that $\ell(I) = 2$, I satisfies G_2 condition and AN_0^- , and $\text{depth}(R/I) \geq \text{Min}\{\dim R/I, 1\}$. Let $J = (x_1, x_2)$ be a general minimal reduction of I and assume there exists a positive integer p such that

- (i) $\lambda(J \cap I^{n+1}/JI^n) = 0$ for every $0 \leq n \leq p-1$.
- (ii) $\lambda(I^{p+1}/JI^p) \leq 1$.

Then

- (a) x_1^* is regular on $\text{gr}_I(R)_+$.
- (b) $\text{depth}(\text{gr}_I(R)) \geq 1$.

Proof. We first prove part (a). If I is \mathfrak{m} -primary then both claims follow from [24, Theorem 4.4]. Thus we may assume that $\dim(R/I) > 0$. Since $\lambda(I^{p+1}/JI^p) \leq 1$, one has $I^{p+1} = (ab) + JI^p$ for some $a \in I, b \in I^p$ with $ab \notin JI^p$. For $n \geq p$, the multiplication by a gives a surjective map from I^{n+1}/JI^n to I^{n+2}/JI^{n+1} . Thus the length $\lambda(I^{n+1}/JI^n) \leq 1$ for every $n \geq p$.

Notice that x_1 is regular on I , since $(0 : x_1) \cap I = 0$ (by [18, Lemma 3.2]). To prove that x_1^* is regular on $\text{gr}_I(R)_+ = \text{gr}_I(I)$, we only need to show $x_1 I \cap I^n I = x_1 I^{n-1} I$ for every $n \geq 1$ by [31, Proposition 2.6] (see also [24, Lemma 1.1]). This is clear if $n = 1$; hence we may assume $n \geq 2$. Let $'$ denote images in $R' = R/(x_1)$ and set $s = r(I')$. We claim that it is enough to show $r(I) = s$. Indeed, if $r(I) = s$, then $x_1 I \cap I^n I = x_1 I \cap J I^{n-1} I$ for every $n \geq 1$. This is clear if $s \leq p$. Assume $s > p$. If $n \leq p - 1$, then $x_1 I \cap I^n I = x_1 I \cap J \cap I^n I = x_1 I \cap J I^{n-1} I$. If $p \leq n \leq s - 1$, then

$$\begin{aligned} 0 &< \lambda(I^n I / J I^{n-1} I + (x_1) \cap I^n I) \\ &= \lambda(I^n I / J I^{n-1} I) - \lambda(J I^{n-1} I + (x_1) \cap I^n I / J I^{n-1} I) \\ &= 1 - \lambda(J I^{n-1} I + (x_1) \cap I^n I / J I^{n-1} I), \end{aligned}$$

which yields $J I^{n-1} I + (x_1) \cap I^n I = J I^{n-1} I$. Furthermore, if $n \geq s = r(I)$, then $I^n I = J I^{n-1} I$ and, therefore, $(x_1) I \cap I^n I = x_1 I \cap J I^{n-1} I$ for every $n \geq s$. Now an argument similar to the one of [18, 4.7] gives $x_1 I \cap I^n I = x_1 I^{n-1} I$ for every $n \geq 1$.

To complete the proof of part (a), we still need to show that $r(I) = s$. This follows by an argument similar to the one employed in [18, 4.7]. We write it for the sake of completeness. We use a result on the Ratliff-Rush filtration $\widetilde{I}^n I := \cup_{t \geq 1} (I^{n+t} I :_I I^t)$ (see [24, Theorem 4.2] or [18, Corollary 4.5]). Since x_1 is regular on I , by [24, Lemma 3.1], there exists an integer n_0 such that $I^n I = \widetilde{I}^n I$ for $n \geq n_0$, and

$$(2) \quad \widetilde{I}^{n+1} I :_I x_1 = \widetilde{I}^n I \quad \text{for every } n \geq 0.$$

On the quotient ring $R' = R/(x_1)$, there are two filtrations:

$$\mathbb{M} : I' \supseteq I'^2 \supseteq \dots \supseteq I'^n \supseteq \dots$$

and

$$\mathbb{N} : I' \supseteq \widetilde{I} I' \supseteq \dots \supseteq \widetilde{I}^{n-1} I' \supseteq \dots$$

Notice that \mathbb{M} is an I' -adic filtration and \mathbb{N} is a good I' -filtration (see [24, page 9] for the definition of good filtrations). Notice $\lambda(I'/I'^2) < \infty$. Since $I^n I' = \widetilde{I}^n I'$ for $n \geq n_0$, the associated graded modules $\text{gr}_{\mathbb{M}}(I')$ and $\text{gr}_{\mathbb{N}}(I')$ have the same Hilbert coefficients e_0 and e_1 . Since I contains a non zero divisor on I' , by [24, Lemmas 2.1 and 2.2], we have

$$\begin{aligned} &\sum_{n \geq 0}^{p-2} \lambda(I^{n+1} I / J I^n I) + (s-1) - (p-2) \\ &= \sum_{n \geq 0} \lambda(I'^{n+2} / J' I'^{n+1}) = e_1(\mathbb{M}) = e_1(\mathbb{N}) = \sum_{n \geq 0} \lambda(\widetilde{I}^{n+1} I' / J \widetilde{I}^n I'). \end{aligned}$$

The first equality follows from the fact that, for $0 \leq n \leq s-1$, one has

$$\lambda(I^{n+1} / J I^n) = \lambda(I'^{n+1} / J' I'^n).$$

Indeed, if $0 \leq n \leq p-1$ then $\lambda(J \cap I^{n+1} / J I^n) = 0$. Therefore,

$$\begin{aligned} \lambda(I^{n+1} / J I^n) &= \lambda(I^{n+1} / J \cap I^{n+1}) = \lambda(I'^{n+1} / J' \cap I'^{n+1}) \\ &\leq \lambda(I'^{n+1} / J' I'^n) \leq \lambda(I^{n+1} / J I^n). \end{aligned}$$

On the other hand, if $p \leq n \leq s-1$, we have $0 < \lambda(I^{n+1}/J'I^n) \leq \lambda(I^{n+1}/JI^n) = 1$. This proves that $\lambda(I^{n+1}/J'I^n) = \lambda(I^{n+1}/JI^n) = 1$ for $p \leq n \leq s-1$ and $\lambda(I^{n+1}/J'I^n) = 0$ for $n \geq s$.

We now prove that $\lambda(\widetilde{I^{n+1}I'}/\widetilde{JI^nI'}) = \lambda(\widetilde{I^{n+1}I}/\widetilde{JI^nI})$ for every $n \geq 0$. Since

$$\widetilde{I^{n+1}I'}/\widetilde{JI^nI'} \cong \widetilde{I^{n+1}I}/((x_1) \cap \widetilde{I^{n+1}I} + x_2\widetilde{I^nI}),$$

we just need to show $(x_1) \cap \widetilde{I^{n+1}I} = (x_1)\widetilde{I^nI}$. We first prove $(x_1) \cap \widetilde{II} = x_1I$. Since $(x_1) \cap \widetilde{II} \supseteq x_1I$, it suffices to show the equality locally at every associated prime ideal of R/x_1I . By Lemma [18, 3.2], every $\mathfrak{p} \in \text{Ass}(R/x_1I)$ is not maximal. Hence $(x_1)_{\mathfrak{p}} = \widetilde{I}_{\mathfrak{p}} = I_{\mathfrak{p}}$ and $(x_1)_{\mathfrak{p}} \cap \widetilde{II}_{\mathfrak{p}} = \widetilde{II}_{\mathfrak{p}} = x_1I_{\mathfrak{p}}$. This shows $(x_1) \cap \widetilde{II} = x_1I$. Now for any $n \geq 1$, $(x_1) \cap \widetilde{I^{n+1}I} = x_1I \cap \widetilde{I^{n+1}I} = x_1(\widetilde{I^{n+1}I} :_I x_1) = x_1\widetilde{I^nI}$. Hence we have

$$(3) \quad \sum_{n \geq 0} \lambda(\widetilde{I^{n+1}I}/\widetilde{JI^nI}) = \sum_{n \geq 0}^{p-2} \lambda(I^{n+1}I/JI^nI) + (s-1) - (p-2).$$

Let $W_J = \{t \in \mathbb{N} \mid \widetilde{JI^nI} \cap I^{n+1}I = JI^nI, 0 \leq n \leq t\}$. Then $p-2 \in W_J$. Hence, by [24, Theorem 4.2], we have

$$r(I) \leq \sum_{n \geq 0} \lambda(\widetilde{I^{n+1}I}/\widetilde{JI^nI}) + p-1 - \sum_{n=0}^{p-2} \lambda(I^{n+1}I/JI^nI) = s.$$

Finally, since $\text{depth}(R/I) > 0$ and $0 \rightarrow R/I \rightarrow \text{gr}_I(R) \rightarrow \text{gr}_I(R)_+ \rightarrow 0$ is exact, by part (a), we have

$$\text{depth}(\text{gr}_I(R)) \geq \min\{\text{depth } R/I, \text{depth}(\text{gr}_I(R)_+)\} \geq 1.$$

□

We can now prove our second main result.

Theorem 4.6. [Sally's Conjecture for j -stretched ideals] Assume R and I satisfy Setting 3.3 (I). Let I be j -stretched. If there exists a positive integer p such that

- (a) $\lambda(J \cap I^{n+1}/JI^n) = 0$ for every $0 \leq n \leq p-1$,
- (b) $\lambda(I^{p+1}/JI^p) \leq 1$,

then

- (i) for a general $x_1 \in I$, x_1^* is regular on $\text{gr}_I(R)_+$.
- (ii) $\text{depth}(\text{gr}_I(R)) \geq d-1$.

Proof. We prove the theorem by induction on d . The case $d = 2$ has been proved in Theorem 4.5. Let $d \geq 3$ and assume the theorem holds for $d-1$. We first reduce to the case of $\text{grade } I \geq 1$. If $\text{grade } I = 0$, let $H_0 = 0 : I$. As in the proof of Theorem 4.1, all assumptions still hold for the quotient ring R/H_0 . Furthermore, $I/H_0 \cap I = I$, $\text{grade}(I/H_0 \cap I) \geq 1$ and $\text{depth}(\text{gr}_I(R)) \geq \text{depth}(\text{gr}_I(R/H_0))$. So we are reduced to the case where the ideal I contains at least one regular element on R . Thus x_1 is regular on R .

If $\dim R/I = 0$ then the assertion follows from [24, Theorem 4.4]. Hence, we may assume $\dim R/I > 0$. Let $'$ denote images in $R' = R/(x_1)$. Observe that R' is a Cohen-Macaulay ring of dimension $d-1$ and $\ell(I') = d-1$. Also, I' satisfies G_{d-1} and AN_{d-3}^- (see [18, Lemma 3.2]). Furthermore, observe that $R'/I' \cong R/I$, whence $\text{depth}(R'/I') = \text{depth}(R/I) \geq \min\{\dim R/I, 1\} = \{\dim R'/I', 1\}$. Clearly, I' is j -stretched in R' . By induction hypothesis, for a general $x_2 \in I$, x_2^* is regular on $\text{gr}'_{I'}(R')_+$, and $\text{depth}(\text{gr}_{I'}(R')) \geq d-2$.

By [18, Lemmas 4.8 and 4.9], one has that x_1^* is regular on $\text{gr}_I(R)$. Since $\text{depth}(\text{gr}_{I'}(R')) \geq d-2$ and x_1^* is regular on $\text{gr}_I(R)$, we have $\text{depth}(\text{gr}_I(R)) \geq d-1$. \square

As an application of Theorem 4.6, we obtain a sufficient condition for the almost Cohen-Macaulayness of the associated graded rings of j -stretched ideals.

Corollary 4.7. *Let R and I be as in Setting 3.3. If I is j -stretched with index of nilpotency K , then*

- (a) $I^{K+1} \subseteq JI^{K-1}$ if and only if $\lambda(I^K/JI^{K-1}) = 1$.
- (b) If $I^{K+1} \subseteq JI^{K-1}$ then $\text{depth}(\text{gr}_I(R)) \geq d-1$.

Proof. Part (a) follows by the same argument as in [23, Proposition 3.1]. From part (a) and Proposition 3.7, one has $\lambda(I^K/JI^{K-1}) \leq 1$ and $\lambda(J \cap I^{n+1}/JI^n) = 0$ for every $0 \leq n \leq K-1$. Hence part (b) follows by applying Theorem 4.6 with $p = K-1$. \square

As a special case of Corollary 4.7, we recover also the second main result of Polini-Xie [18].

Corollary 4.8. ([18, Theorem 4.10]) *Let R be a d -dimensional Cohen-Macaulay local ring and let I be an ideal with $\ell(I) = d$, $\text{depth}(R/I) \geq \min\{\dim(R/I), 1\}$ and I satisfies G_d and AN_{d-2}^- . If I has almost minimal j -multiplicity, then $\text{depth}(\text{gr}_I(R)) \geq d-1$.*

Proof. If I has almost minimal j -multiplicity then $K = 2$. Since $I^2 \not\subseteq J$ and $\lambda(I^2/IJ) = 1$, one has $I^2 \cap J = IJ$. Therefore, $I^3 \subseteq JI$. Now Corollary 4.7 finishes the proof with $K = 2$. \square

In [20] and [23], it was introduced the concept of type of an ideal I with respect to a given minimal reduction J of I . This was defined as $\tau(I) = \lambda((J : I) \cap I/J)$, a number that depends heavily on the choice of J . Here we introduce a slight variation of this concept that fits with our setting. For a *general* minimal reduction J of I , we set

$$\tau(I) = \lambda((J : I) \cap I/J),$$

and call it the *general Cohen-Macaulay type* of I . It follows immediately from the Specialization Lemma (3.1) that, in presence of the G_d condition, this number is well-defined, because it is constant for J general.

Lemma 4.9. *Assume R is Cohen-Macaulay. Let I be an ideal having $\ell(I) = d$ and the G_d condition. Then the number $\tau(I)$ is independent of the general minimal reduction J .*

In the same spirit of the definitions given in [18], we say that an ideal I has *almost almost minimal* j -multiplicity if $\lambda(\overline{I^2}/x_d\overline{I}) \leq 2$, or equivalently, if $K \leq 3$.

Next we want to prove that the associated graded rings of j -stretched ideals having almost-almost minimal j -multiplicity (i.e. $K = 3$) and small general Cohen-Macaulay type are almost Cohen-Macaulay. This provides a higher dimensional version of results of [23]. The first step in this direction consists in proving that j -stretched ideals of small general type satisfy the inclusion $I^{K+1} \subseteq JI^2$. Recall the embedding codimension of I is defined as $h(I) = \lambda(\overline{I}/\overline{I}^2) - \lambda(\overline{R}/\overline{I})$.

Theorem 4.10. *Assume R and I satisfy Setting 3.3 (1) and (2). Let I be j -stretched with $K = s(I)$. Let $J = (x_1, \dots, x_d)$ be a general minimal reduction of I and set $\overline{R} = R/J_{d-1} : I^\infty$, where $J_{d-1} = (x_1, \dots, x_{d-1})$. If $\tau(I) < h(I) + 1 - \lambda(\overline{R}/\overline{I})$, then*

$$\nu_2 = K - 2, \quad J \cap I^3 = JI^2.$$

In particular, $I^{K+1} \subseteq JI^2$.

Proof. Similar to the proof of [23, Theorem 2.7]. \square

The next theorem generalizes several classical results, see for instance [26], [22], [23], [24] and [18].

Corollary 4.11. *Assume R and I satisfy Setting 3.3 (1) and (2). Let I be j -stretched with $K = s(I)$ and let h be the embedding codimension of I as in Theorem 4.10. If either (i) $K \leq 2$, or (ii) $K = 3$ and $\tau(I) < h(I) + 1 - \lambda(\overline{R}/\overline{I})$, then*

$$\text{depth}(\text{gr}_I(R)) \geq d - 1.$$

Proof. Since the cases $K = 1, 2$ have been proved in [18], we only need to prove the case $K = 3$. By Theorem 4.10, we have that $I^4 \subseteq JI^2$. Corollary 4.7 now finishes the proof. \square

We conclude this section with the example of an ideal I having minimal j -multiplicity, not having G_d condition and for which $\text{gr}_I(R)$ is not Cohen-Macaulay. It demonstrates that the residual assumptions in our main Theorems are necessary.

Example 4.12. (see [4] or [18, 3.10]) *Let $R = k[[x, y, z]]/(x^3 - x^2y)$ and $I = (xy^t, z)$ for any $t \geq 0$. Then R is a two-dimensional Cohen-Macaulay local ring, $\ell(I) = 2$, and I has reduction number zero. In particular I has minimal j -multiplicity. However, I does not satisfy G_2 , and $\text{gr}_I(R)$ is not Cohen-Macaulay.*

5. THE \mathfrak{m} -PRIMARY CASE

In this section we prove the non trivial fact that j -stretched ideals (strictly) generalize the stretched \mathfrak{m} -primary ideals introduced by Sally and Rossi-Valla. First, recall that an \mathfrak{m} -primary ideal I is said to be *stretched* if there exists a minimal reduction H of I such that

$$(a) \quad H \cap I^2 = HI.$$

$$(b) \ HF_{I/H}(2) \leq 1.$$

This definition, first given in [23], extends the classical concept of stretched Cohen-Macaulay local rings given by Sally in [26]. If R is Cohen-Macaulay, stretched \mathfrak{m} -primary ideals include ideals having minimal multiplicity (see for instance [24]). However, there are \mathfrak{m} -primary ideals with almost minimal multiplicity that are not stretched, even in 1-dimensional Cohen-Macaulay local rings. In contrast, j -stretched ideals include ideals having minimal or almost minimal multiplicity, because they include ideals of minimal and almost minimal j -multiplicity.

We first prove that general minimal reductions always achieve the minimal colength.

Proposition 5.1. *Let I be an ideal which has $\ell(I) = d$ and the G_d condition. Let H and J be a minimal and a general minimal reduction of I , respectively. Let $n \geq 1$ be a fixed integer. Then the lengths $\lambda(I^n/J^n)$ and $\lambda(I^n/JI^{n-1} + I^{n+1})$ do not depend on J . Furthermore, if R is equicharacteristic, one has*

- (a) $\lambda(I^n/J^n) \leq \lambda(I^n/H^n)$.
- (b) $\lambda(I^n/JI^{n-1} + I^{n+1}) \leq \lambda(I^n/HI^{n-1} + I^{n+1})$.

Proof. Let \mathfrak{m} be the maximal ideal of R and write $I = (a_1, \dots, a_s)$. To prove assertion (a), take $d \times s$ variables, say $\underline{z} = (z_{ij})$, and set $S = R[\underline{z}]$, $J' = (x'_1, \dots, x'_d)S$, where $x'_i = \sum_{j=1}^s z_{ij}a_j$, $1 \leq i \leq d$. Let $\underline{\alpha}_0 \in R^{ds}$ be the vector such that $J'_{\underline{\alpha}_0} = H$. Since I has the G_d condition, we have $\lambda_{S_{\mathfrak{m}S}}(IS_{\mathfrak{m}S}/J'S_{\mathfrak{m}S}) < \infty$. By Lemma 3.1, for a general element $\underline{\alpha} \in R^{ds}$, we have

$$\lambda(I^n/J^n) = \lambda(I^n/[(J')^n S]_{\underline{\alpha}}) = \lambda_{S_{\mathfrak{m}S}}(I^n S_{\mathfrak{m}S}/(J')^n S_{\mathfrak{m}S}).$$

Furthermore, if R is equicharacteristic, we have $\lambda(I^n/J^n) \leq \lambda(I^n/[(J')^n S]_{\underline{\alpha}_0}) = \lambda(I^n/H^n)$. Assertion (b) can be proved similarly. \square

We can then compare the lengths of quotients that are relevant for stretched ideals.

Proposition 5.2. *Let (R, \mathfrak{m}) be a d -dimensional equicharacteristic Cohen-Macaulay local ring with infinite residue field. Let I be an \mathfrak{m} -primary ideal, and H a minimal reduction of I . Then for a general minimal reduction J of I , one has*

$$\lambda((J \cap I^2)/JI) \leq \lambda((H \cap I^2)/HI).$$

In particular, if $H \cap I^2 = HI$ then one has $J \cap I^2 = JI$.

Proof. For any ideal $L \subseteq I$, we have $\lambda((L \cap I^2)/LI) = \lambda(I^2/LI) - \lambda(I^2/L \cap I^2)$, and $\lambda(I^2/L \cap I^2) = \lambda(R/L) - \lambda(R/I^2 + L)$, so that we obtain

$$(4) \quad \lambda((L \cap I^2)/LI) = \lambda(I^2/LI) - \lambda(R/L) + \lambda(R/(I^2 + L)).$$

Observe that

- $\lambda(I^2/JI) = \lambda(I^2/HI)$ (see, for instance, [24, Corollary 2.1]).
- $\lambda(R/J) = e(R) = \lambda(R/H)$, because J and H are minimal reductions of I .

- $\lambda(R/I^2 + J) \leq \lambda(R/I^2 + H)$, by Lemma 3.1.

Together with equation (4), the above gives $\lambda((J \cap I^2)/JI) \leq \lambda((H \cap I^2)/HI)$. \square

We now prove the main result of this section. It shows that, in certain situations, j -stretchedness can be checked by using a *special* minimal reduction (instead of every *general* minimal reduction). In particular, it gives a concrete criterion to construct examples of j -stretched ideals.

Theorem 5.3. *Let (R, \mathfrak{m}) be a d -dimensional equicharacteristic Cohen-Macaulay local ring with infinite residue field. let I be an ideal with $\ell(I) = d$. Let $H = (y_1, \dots, y_d)$ be a minimal reduction of I . Set $H_{d-1} = (y_1, \dots, y_{d-1})$ and assume*

$$\lambda(I^2/[y_d I + I^3 + (H_{d-1} : I^\infty) \cap I^2]) \leq 1.$$

If one of the following two conditions holds,

- (i) *I is \mathfrak{m} -primary and $H \cap I^2 = HI$,*
- (ii) *$\text{depth}(R/I) \geq 1$, I has properties G_d and AN_{d-2}^- , and $H_{d-1} : I$ is a geometric $d - 1$ -residual intersection of I ,*

then I is j -stretched.

Proof. It suffices to show that either (i) or (ii) implies the equality

$$y_d I + I^3 + (H_{d-1} : I^\infty) \cap I^2 = HI + I^3.$$

Note that, to prove the above, one does not need the inequality $\lambda(I^2/[y_d I + I^3 + (H_{d-1} : I^\infty) \cap I^2]) \leq 1$.

First assume (i) holds. Since R is Cohen-Macaulay, then I contains a non zero divisor on R/H_{d-1} , whence $H_{d-1} : I^\infty = H_{d-1}$. One then has

$$(H_{d-1} : I^\infty) \cap I^2 = H_{d-1} \cap I^2 = H_{d-1} \cap H \cap I^2 = H_{d-1} \cap HI.$$

Now, we have

$$\begin{aligned} H_{d-1} \cap HI &= H_{d-1} \cap (H_{d-1}I + y_d I) = H_{d-1}I + (H_{d-1} \cap y_d I) \\ &= H_{d-1}I + y_d(H_{d-1} :_I y_d) = H_{d-1}I + y_d H_{d-1} \\ &= H_{d-1}I, \end{aligned}$$

showing that $(H_{d-1} : I^\infty) \cap I^2 = H_{d-1}I$, which immediately implies $y_d I + I^3 + (H_{d-1} : I^\infty) \cap I^2 = HI + I^3$.

Next, assume (ii) holds. Since $H_{d-1} : I$ is a geometric $d - 1$ -residual intersection of I and I satisfies AN_{d-2}^- , we have $H_{d-1} : I^\infty = H_{d-1} : I$. This time the equality $(H_{d-1} : I^\infty) \cap I^2 = H_{d-1}I$ follows by an argument similar to [18, Lemma 3.2].

Then, in either case, one has $\lambda(I^2/HI + I^3) \leq 1$. By Lemma 3.1, this implies $\lambda(I^2/JI + I^3) \leq 1$ for a general minimal reduction J of I , showing that I is j -stretched. \square

As a consequence, we immediately obtain that every stretched \mathfrak{m} -primary ideal is j -stretched.

Corollary 5.4. *Let (R, \mathfrak{m}) be an equicharacteristic Cohen-Macaulay local ring with infinite residue field and I an R -ideal. If I is a stretched \mathfrak{m} -primary ideal then I is a j -stretched ideal.*

Proof. By the first half of the definition of stretched \mathfrak{m} -primary ideals I , these ideals satisfy assumptions (i) of Theorem 5.3. As in the proof of Theorem 5.3, we then have the equality

$$[y_d I + I^3 + (H_{d-1} : I^\infty) \cap I^2] = HI + I^3$$

In particular, we have $\lambda(I^2/[y_d I + I^3 + (H_{d-1} : I^\infty) \cap I^2]) = \lambda(I^2/HI + I^3) = \lambda(I^2/H \cap I^2 + I^3) = HF_{I/H}(2)$. By assumption of stretchedness, this length is at most 1, proving that stretched ideals satisfy the inequality required in Theorem 5.3. We can then apply Theorem 5.3 to conclude that I is j -stretched. \square

One may wonder if, in the \mathfrak{m} -primary case, j -stretchedness coincides with stretchedness. In general, this is not the case. For instance, the ideal $I = (t^3, t^4)$ in the ring $A = k[[t^3, t^4, t^5]]$ is j -stretched, has almost minimal multiplicity, but is not stretched. This shows that j -stretched \mathfrak{m} -primary ideals strictly generalize classical stretched ideals, even in the 1-dimensional case.

In contrast, we now provide a condition ensuring that stretchedness coincides with j -stretchedness.

Proposition 5.5. *Let (R, \mathfrak{m}) be an equicharacteristic Cohen-Macaulay local ring and I an \mathfrak{m} -primary ideal. Assume $I^2 \cap H = HI$ for a minimal reduction H of I . Then I is stretched if and only if I is j -stretched.*

Proof. By Corollary 5.4 we only need to show that, if I is j -stretched, then I is stretched. Let $J = (x_1, \dots, x_d)$ be a general minimal reduction of I and $J_{d-1} = (x_1, \dots, x_{d-1})$. By j -stretchedness we have $\lambda(\bar{I}^2/(x_d \bar{I} + \bar{I}^3)) \leq 1$, where $\bar{R} = R/(J_{d-1} : I^\infty) = R/J_{d-1}$. By Proposition 5.2, one obtains $J \cap I^2 = JI$. Hence, we get

$$\bar{I}^2/(x_d \bar{I} + \bar{I}^3) \cong I^2/(JI + I^3 + J_{d-1} \cap I^2) = I^2/(JI + I^3) = I^2/((J \cap I^2) + I^3).$$

Therefore, for a general minimal reduction J of I , one has $HF_{I/J}(2) \leq 1$. This fact, together with $J \cap I^2 = JI$, proves the stretchedness of I . \square

We conclude this section with an application of the above results to answer a question of Sally. If I is \mathfrak{m} -primary, classical examples by Sally and Rossi-Valla show that I can be stretched with respect to a minimal reduction J_1 but not stretched with respect to a different minimal reduction J_2 . Hence it is well known that the stretchedness property depends upon the minimal reduction. Sally [26] raised the following question: *To what extent does the concept of “stretchedness” depend upon the choice of the minimal reduction?* We are now able to answer this question.

Corollary 5.6. *Let (R, \mathfrak{m}) be an equicharacteristic Cohen-Macaulay local ring and I an \mathfrak{m} -primary ideal. If I is stretched with respect to a minimal reduction H , then I is stretched with respect to any general minimal reduction.*

Proof. Let J be a general minimal reduction of I . By Proposition 5.2, the “intersection property” $J \cap I^2 = JI$ follows at once from $H \cap I^2 = HI$. Then we only need to show that $\lambda(I^2/(J \cap I^2) + I^3) \leq 1$. By Proposition 5.1 we have $\lambda(I^2/JI + I^3) \leq \lambda(I^2/HI + I^3)$. We have then obtained the following chain of inequalities

$$\lambda(I^2/(J \cap I^2) + I^3) = \lambda(I^2/JI + I^3) \leq \lambda(I^2/HI + I^3) = \lambda(I^2/(H \cap I^2) + I^3) \leq 1.$$

□

The examples of Sally and Rossi-Valla show that I is stretched with respect to a minimal reduction does not imply that I is stretched with respect to every minimal reduction of I . However, Corollary 5.6 proves that the next best possible scenario holds, that is, I is stretched with respect to a Zariski dense open subset of minimal reductions of I .

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